

On the application of rational expression in S_b metric space

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Abstract

In this paper, we investigate the result of fixed point in very recently defined ε -chainable S_b -metric space by using rational expression. Our result generalize and extend the theorem 3.3 of Pagey and Malviya [9] in ε -chainable S_b metric space [7].

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Fixed point theorem.

1- Introduction and Preliminaries

In 2012, Sedghi et al.[11] defined a generalization of G metric space which is called S- metric space. In 2016, Souayan N. et al. [13]introduced the S_b -metric space as a generalization of the b-metric space and S metric Space, and proved some fixed point results under different types of contractions in a complete S_b -metric space. In 2017, Rohen Y et al. [10] modified the definition of S_b -metric and prove some coupled common fixed point theorems in S_b -metric space.

On the other hand, as long back in 1983 Cantor defined connectedness with the help of ε -chains which have been studied extensively by many mathematicians [see Reference 1,2,3,4,5,6,8].

Very recently Malviya.[7] defined ε -chain in S_b -metric space with examples and proved a fixed point theorem using Hardy Rogers contraction condition. In this paper, after introduction and preliminaries in section 2, we have proved a fixed point theorem using rational expression. Our result extends and generalizes the theorem 3.3. of Pagey et.al.[9]

Definition 1.1[10]: Let X be a nonempty set and $s \geq 1$ be a given real number. An S_b - metric on X is a function $S_b: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S_b(x, y, z) = 0$ if and only if $x = y = z$
2. $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$

Then the function S_b is called an S_b - metric and the pair (X, S_b) is called an S_b -metric space.

Definition 1.2[10]. Let (X, S_b) be a S_b -metric space then S_b is called symmetric if

$$S_b(x, x, y) = S_b(y, y, x) \quad \forall x, y \in X \text{ (Symmetric Property)}$$

For the definitions of convergence and Cauchy sequence in S_b metric space reader can refer [10,12 and 13]

Definition 1.3 [7]:- Let (X, S_b) be a S_b -metric space. An ε - chain is a finite succession of points $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ in X such that $S_b(a_{i-1}, a_{i-1}, a_i) \leq \varepsilon$ for $i = 1, 2, \dots, n$. The integer n is called the length of the ε - chain

Definition 1.4[7] :- An space (X, S_b) is ε -chainable (ε -connected) if every pair of points in it can be joined by an ε -chain of points in the set X and (X, S_b) is called chainable if it is ε -chainable for each positive ε .

The space (X, S_b) is called complete chainable S_b -metric space if every Cauchy sequence converges in it.

Throughout this paper length of ε -chain between any two points x and y in X means length of shortest ε -chain between points x and y in X .

Example 1.5[7] $X=A \cup B$ and S_b be the metric on $X \times X \times X$ to R^+ defined by

$$S_b(x, y, z) = (|x - z| + |y - z|)^2 \text{ where}$$

$$A = \{1, 1/2, 1/3\} \text{ and } B = \{1/2^2, 1/3^2, 1/4^2\}$$

Then Y is ε -chainable for $\varepsilon = 1$ and length of biggest ε -chain in X is 5.

Example 1.6 [7] : Let (R, S_b) be the S_b - metric space as defined in Example 1.5 and $Y = A \cup B$ where $A = [1, 2)$ and $B = (2, 3]$. Then (Y, S_b) is a disconnected subspace of (R, S_b) but is ε -chainable for every $\varepsilon > 0$. If $\varepsilon = 1$ then length of biggest ε -chain in space (Y, S_b) is 5.

2.Main Result: The following result provides the application of rational inequality to find fixed

point in S_b metric space

Theorem 2.1: Let (X, S_b) be a complete ε -chainable S_b -metric space and let f, g be two self maps of X , if

$$\varepsilon = S_b(fx, fx, gy) \leq c \left[\frac{S_b(y, y, fx)^2 + S_b(x, x, gy)^2}{S_b(y, y, fx) + S_b(x, x, gy)} \right] \dots \dots \dots (1)$$

for all $x, y \in X$ where a_1, a_2, a_3, a_4, a_5 are non negative reals. And $S_b(y, y, fx) + S_b(x, x, gy) \neq 0$

$$0 \leq Msc < \frac{1}{3} \text{ and } c < 1 \dots \dots \dots (2)$$

where M is the length of the biggest ε -chain in space (X, S_b) such that $M \geq 0$. Then f and g have unique common fixed point.

Proof. We define a sequence $\{x_n\}$ as follows

$$fx_{2n+1} = x_{2n} \text{ and } gx_{2n+2} = x_{2n+1} \text{ for } n = 0, 1, 2, 3 \dots$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n , then we see that x_{2n} is a fixed point of f and g therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal. If N is the length of ε -chain between x_{2n+1} and x_{2n+2} then ε -chainability of (X, S_b) gives.

$$\begin{aligned} S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) &\leq N\varepsilon \\ &\leq M\varepsilon \quad \text{as } M \geq N \\ &= MS_b(fx_{2n+1}, fx_{2n+1}, gx_{2n+2}) \quad \text{using (1)} \\ &\leq Mc \left[\frac{S_b(x_{2n+2}, x_{2n+2}, fx_{2n+1})^2 + S_b(x_{2n+1}, x_{2n+1}, gx_{2n+2})^2}{S_b(x_{2n+2}, x_{2n+2}, fx_{2n+1}) + S_b(x_{2n+1}, x_{2n+1}, gx_{2n+2})} \right] \\ &= Mc \left[\frac{S_b(x_{2n+2}, x_{2n+2}, x_{2n})^2 + S_b(x_{2n+1}, x_{2n+1}, x_{2n+1})^2}{S_b(x_{2n+2}, x_{2n+2}, x_{2n}) + S_b(x_{2n+1}, x_{2n+1}, x_{2n+1})} \right] \\ &= Mc [S_b(x_{2n+2}, x_{2n+2}, x_{2n})] \\ &\leq Msc [S_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_b(x_{2n}, x_{2n}, x_{2n+1})] \\ &= Msc [S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S_b(x_{2n}, x_{2n}, x_{2n+1})] \end{aligned}$$

$$(1 - 2Msc)S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq Msc[S_b(x_{2n}, x_{2n}, x_{2n+1})]$$

$$S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \frac{Msc}{(1-2Msc)} [S_b(x_{2n}, x_{2n}, x_{2n+1})]$$

In general

$$S_b(x_n, x_n, x_{n+1}) \leq \frac{Msc}{(1-2Msc)} [S_b(x_{n-1}, x_{n-1}, x_n)]$$

$$\Rightarrow S_b(x_n, x_n, x_{n+1}) \leq k[S_b(x_{n-1}, x_{n-1}, x_n)] \text{ where } k = \frac{Msc}{(1-2Msc)}$$

$$\Rightarrow S_b(x_n, x_n, x_{n+1}) = k^n S_b(x_0, x_0, x_1) \text{ for } n = 1, 2, 3 - -$$

Now we prove that $\{x_n\}$ is a Cauchy sequence, if p is any positive integer then we have.

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sS_b(x_n, x_n, x_{n+1}) + sS_b(x_{n+p}, x_{n+p}, x_{n+1}) \text{ by 2 of definition 1.1}$$

$$= 2sS_b(x_n, x_n, x_{n+1}) + sS_b(x_{n+1}, x_{n+1}, x_{n+p}) \text{ by symmetric property}$$

$$\leq 2sS_b(x_n, x_n, x_{n+1}) + 2s^2S_b(x_{n+1}, x_{n+1}, x_{n+2}) + s^2S_b(x_{n+p}, x_{n+p}, x_{n+2})$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sS_b(x_n, x_n, x_{n+1}) + 2s^2S_b(x_{n+1}, x_{n+1}, x_{n+2})$$

..... +

$$2s^{n+p}S_b(x_{n+p-1}, x_{n+p-1}, x_{n+p}).$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sk^n S_b(x_0, x_0, x_1) + 2s^2k^{n+1} S_b(x_0, x_0, x_1)$$

$$..... + 2s^{n+p}k^{n+p-1} S_b(x_0, x_0, x_1)$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq (2sk^n + 2s^2k^{n+1} + \dots + 2s^{n+p}k^{n+p-1})S_b(x_0, x_0, x_1)$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) < \frac{2sk^n}{(1-sk)} S_b(x_0, x_0, x_1)$$

$$\text{As } n \rightarrow \infty, S_b(x_n, x_n, x_{n+p}) \rightarrow 0$$

Therefore $\{x_n\}$ is a Cauchy sequence in X . As X is a complete space so there exists a point x in X such that $\lim_{n \rightarrow \infty} x_n = x$.

Existence of fixed point :

Consider

$S_b(x, x, gx) \leq 2S_b(x, x, x_{2n}) + S_b(x_{2n}, x_{2n}, gx)$ by Definition 1.2 and 2 of definition 1.1

$$\begin{aligned} \Rightarrow S_b(x, x, gx) &\leq 2S_b(x, x, x_{2n}) + S_b(fx_{2n+1}, fx_{2n+1}, gx) \\ &\leq 2S_b(x, x, x_{2n}) + \\ c \left[\frac{S_b(x, x, fx_{2n+1})^2 + S_b(x_{2n+1}, x_{2n+1}, gx)^2}{S_b(x, x, fx_{2n+1}) + S_b(x_{2n+1}, x_{2n+1}, gx)} \right] \\ &\leq 2S_b(x, x, x_{2n}) + \\ c \left[\frac{S_b(x, x, fx_{2n+1})^2 + S_b(x_{2n+1}, x_{2n+1}, gx)^2}{S_b(x, x, fx_{2n+1}) + S_b(x_{2n+1}, x_{2n+1}, gx)} \right] \\ &= 2S_b(x, x, x_{2n}) + c \left[\frac{S_b(x, x, x_{2n})^2 + S_b(x_{2n+1}, x_{2n+1}, gx)^2}{S_b(x, x, x_{2n}) + S_b(x_{2n+1}, x_{2n+1}, gx)} \right] \end{aligned}$$

As $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are subsequences of $\{x_n\}$, as $n \rightarrow \infty, x_{2n+1} \rightarrow x, x_{2n} \rightarrow x$

as $n \rightarrow \infty, x_{2n} \rightarrow x$ and $x_{2n+1} \rightarrow x$

$$\begin{aligned} S_b(x, x, gx) &\leq cS_b(x, x, gx) \\ \Rightarrow (1 - c)S_b(x, x, gx) &\leq 0 \quad (\text{as } c < 1) \\ \Rightarrow S_b(x, x, gx) &= 0 \\ \Rightarrow gx &= x \end{aligned}$$

Again

$S_b(x, x, fx) \leq 2S_b(x, x, x_{2n+1}) + S_b(fx, fx, gx_{2n+2})$ by Definition 1.2 and 2 of definition 1.1

$$\begin{aligned} \Rightarrow S_b(x, x, gx) &\leq 2S_b(x, x, x_{2n+1}) + S_b(fx, fx, gx_{2n+2}) \\ &\leq 2S_b(x, x, x_{2n+1}) + c \left[\frac{S_b(x_{2n+2}, x_{2n+2}, fx)^2 + S_b(x, x, gx_{2n+2})^2}{S_b(x_{2n+2}, x_{2n+2}, fx) + S_b(x, x, gx_{2n+2})} \right] \\ &= 2S_b(x, x, x_{2n+1}) + c \left[\frac{S_b(x_{2n+2}, x_{2n+2}, fx)^2 + S_b(x, x, x_{2n+1})^2}{S_b(x_{2n+2}, x_{2n+2}, fx) + S_b(x, x, x_{2n+1})} \right] \end{aligned}$$

As $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ are subsequences of $\{x_n\}$, as $n \rightarrow \infty, x_{2n+1} \rightarrow x, x_{2n+2} \rightarrow x$

$$\begin{aligned} S_b(x, x, fx) &\leq cS_b(x, x, fx) \\ \Rightarrow (1 - c)S_b(x, x, fx) &\leq 0 \quad (\text{as } c < 1) \\ \Rightarrow S_b(x, x, fx) &= 0 \\ \Rightarrow fx &= x \end{aligned}$$

Uniqueness:- Let u be another fixed point of f and g

$$\begin{aligned}
 S_b(fx, fx, gu) &\leq c \left[\frac{S_b(u, u, fx)^2 + S_b(x, x, gu)^2}{S_b(u, u, fx) + S_b(x, x, gu)} \right] \\
 \Rightarrow S_b(fx, fx, gu) &\leq c \left[\frac{S_b(u, u, x)^2 + S_b(x, x, u)^2}{S_b(u, u, x) + S_b(x, x, u)} \right] \\
 &\Rightarrow S_b(u, u, x) \leq c S_b(u, u, x) \\
 &\Rightarrow (1 - c) S_b(u, u, x) \leq 0 \\
 &\Rightarrow S_b(u, u, x) = 0 \quad \text{as } c < 1 \\
 &\Rightarrow u = x
 \end{aligned}$$

This complete the proof.

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References:

1. Beer, G., Which connected metric spaces are compact? Proc. Amer. Math. Soc. Volume 83, Number 4, 1981.
2. Bellamy, D.P., Short paths in homogeneous continua Topology Appl.26 (1987) 287-291.
3. Bing, R.H., Concerning hereditarily indecomposable continua, pacific J. Math. (1951) 43-51
4. Newman, M.H.A. Elements of Topology of plane sets of points Cambridge univ. Press, New york, 1961
5. Naimpally S.A and Pareek C.M., on the compactness of connected sets (preprint).
6. Kiran Shrivastava and Geeta Agrawal, Characterization of ε -chainable sets in metric spaces Indian j. Pure apl. Math 33(6): 933-940 June 2002.
7. Malviya N. The invention of ε - chainable S_b metric space with application, Shodh Prakashan , Vol.3, Issue 12, 10-15, June 2026
8. MINC, P., A fixed point theorem for weakly chainable plane. Continua., transactions of the American. Mathematical Society, Volume 317, Number1 , January 1990.
9. Pagey S.S., Malviya N. "A chain of results on fixed point in an ε -- chainable metric space""Journal of pure mathematics" University of Calcutta. Vol.26, , PP-67-78 , 2009.
10. Rohen Y., Dosenovic T., Radenovic S., A Note on the Paper " A Fixed Point Theorems in S_b -Metric Space Filomat 31:11(2017) , 3335-3346 , <https://doi.org/10.2298/FIL1711335R>

11. Sedghi S, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces Mat. Vesnik 64,3(2012),258-266.
12. Sedghi S., Gholidahneh A. , Dosenovic T., Esfahani J, Radenovic S., Common Fixed Point of Four Maps in S_b - Metric Spaces, Journal of Linear and Topological Algebra , Vol 05, No. 02, 2016,93-104.
13. Souayan N, Mlaiki N., A Fixed Point Theorem in S_b - Metric Spaces, J.Math. Comput. Sci Vol-16 , Issue 2 (2016), 131-139.<http://dx.doi.org/10.22436/jmcs.016.02.01>