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On the Asymptotically Regular Maps in N_b-Fuzzy Metric Space with Applications

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Abstract: - In this paper, we define asymptotically regular map and asymptotically regular sequence in N_b-fuzzy metric space and proved a fixed point theorem on the structure of N_b-fuzzy metric space . This result extend and generalize theorem 2.1 of Goswami et. al. [6]

Keywords: N_b- fuzzy metric space, fixed point theorem, asymptotically regular maps, asymptotically regular sequence.

AMS subject classifications: 47H10; 54H25

Introduction and Preliminaries

In the histry of mathematical analysis there are various generalizations of metric spaces and fuzzy metric spaces. Some of important generalizations are given in [1,3,4,5,7,8,9,10,11,12,13,14,15,16,17]. In 2015, Malviya N. [9] introduced the notion of N-fuzzy metric space, pseudo N-fuzzy metric space and describe some of their properties and examples. Very recently in 2022, Fernandez et. al. [2] defined N_b-fuzzy metric space, quasi N_b-fuzzy metric space and pseudo N_b-fuzzy metric spaces. They proved various theorems related to convergence of sequences and analyze topology of symmetric N_b-fuzzy metric spaces

In this paper, we introdues the notion of asymptotic regularity of mapping and sequence and prove fixed point theorems in N_b-fuzzy metric space.

DEFINITION 1.1[19]:-A mapping $*: [0,1] \times [0,1] \times [0,1] \to [0,1]$is called a continuous t-norm if ([0,1], $*$)is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ for $a \leq c, b \leq d$. Examples of t-norms are $a * b = \min\{a, b\}$, $a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$

DEFINITION 1.2[2]:- A triplet (X, N_b, *, k) is an N_b-fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm, $k \geq 1$ is a real number and N_b is a fuzzy set on $X^3 \times (0,\infty)$satisfying the following conditions for all $x, y, z, a, \in X$ and $r, s, t > 0$

(1) $N_b(x, y, z, t) > 0$
(2) $N_b(x, y, z, t) = 1$ if and only if $x = y = z$
(3) $N_b(x, y, z, k(r+s+t)) \geq N(x, x, a, r) * N(y, y, a, s) * N(z, z, a, t)$
(4) $N_b(x, y, z, .): (0,\infty) \to (0,1)$ is a continuous function.

For other definitions related to N_b Fuzzy metric Space reader can refer [2]

DEFINITION 1.3[20]:- A mapping $\emptyset : [0,1] \to [0,1]$ is called an altering distance function if

(i) φ is strictly decreasing and left continuous.
(ii) φ(λ) = 0 if and only if λ = 1
i.e, $\lim_{\emptyset \to 1-} \emptyset(1) = 0$.

Main Results :

DEFINITION 2.1 :-Let p and q be self mappings on a N_b- fuzzy metric space (X, N_b, *, k) and $\{x_n\}$ be a sequence in X. p is said to be asymptotically regular at a point $x_0 \in$ X if

$$\left(\lim_{n\to\infty} N_b\ (p^n\,(x_0), p^n(x_0), p^{n+1}(x_0), t\right)) = 1, \forall\ t > 0.$$

Also the sequence $\{x_n\}$ is said to be asymptotically regular with respect to the pair (p, q) if

$$\lim_{n\to\infty} N_b(p\,(x_n), p(x_n), q(x_n), t) = 1, \forall\ t > 0.$$

THEOREM 2.2 Let (X, N_b, *, k) be a complete symmetric N_b fuzzy metric space, $\emptyset$ be the altering distance function and p: X → X be such that the following condition is satisfied:

$$\emptyset\big(N_b(p(x), p(x), p(y), t)\big) \leq\ b_1(x,y)\theta\big[\min\{\emptyset\big(N_b(x, x, p(x), t)\big), \emptyset\big(N_b(y, y, p(y), t)\big)\}\big] + b_2(x,y)\psi\big[\emptyset\big(N_b(x, x, p(x), t)\big). \emptyset\big(N_b(y, y, p(y), t)\big)\big] + b_3(x,y)\emptyset\big(N_b(x, x, y, t)\big) + b_4(x,y)\Big(\emptyset\big(N_b(x, x, p(x), t)\big) + \emptyset\big(N_b(y, y, p(y), t)\big)\Big) + b_5(x,y)\big[\emptyset\big(N_b(x,\ x, p(y), t)\big) + \emptyset\big(N_b(p(x), p(x), y, t)\big)\big] \qquad \text{-----------------} \ (1)$$

$\forall\ x, y\ \in X\ , t > 0$ where $b_i : X \times X \to [0,\infty)$, $i = 1, 2, 3\ , 4, 5$ *are such that* for some arbitrarily fixed $0 < \lambda_1 < 1$

$$b_3(x,y) + b_4(x,y) + 2b_5(x,y) \leq \lambda_1$$

And $\theta, \psi : R^+ \to R^+$ are continuous functions at 0 and $\theta(0) = \psi(0) = 0$.

If p is asymptotically regular at some point $x_0 \in X$, then p has a unique fixed point in X.

Proof. Suppose that $\{x_n\}$ is a sequence in X where $x_0 \in X$ and $x_{n+1} = p(x_n)$ ∀ n ≥ 0, Now if for some n ≥ 0, $x_n = x_{n+1}$, then x_n is a fixed point of p . Suppose that $x_n \neq x_{n+1} \forall n$. We show that the sequence $\{x_n\}$ is Cauchy.

Suppose to the contrary $\exists\ 0 < \epsilon < 1, t > 0$ and two sequence of integers $\{r_n\}$ and $\{s_n\}$ such that $r_n > s_n > n$,

$$N_b(x_{r_n}, x_{r_n}, x_{s_n}, t) \leq 1 - \epsilon$$

$$N_b(x_{r_n-1}, x_{r_n-1}, x_{s_n-1}, t) > 1 - \epsilon$$

$$N_b(x_{r_n-1}, x_{r_n-1}, x_{s_n}, t) > 1 - \epsilon,\ \forall n \in \mathbb{N} \cup \{0\} \qquad \text{----------}\ (4)$$

Now we have

$$\begin{aligned} 1 - \epsilon \geq\ & N_b(x_{r_n}, x_{r_n}, x_{s_n}, t) \\ & \geq N_b\left(x_{r_n}, x_{r_n}, x_{r_n-1}, \frac{t}{3k}\right) * N_b\left(x_{r_n}, x_{r_n}, x_{r_n-1}, \frac{t}{3k}\right) \\ & * N_b\left(x_{s_n}, x_{s_n}, x_{r_n-1}, \frac{t}{3k}\right) \end{aligned}$$

$$\Rightarrow 1 - \epsilon \geq \lim_{n\to\infty} N_b(x_{r_n}, x_{r_n}, x_{s_n}, t) \geq (1 * 1 * 1 - \epsilon)$$

(Since p asymptotically regular at x_0)

$$\Rightarrow \lim_{n\to\infty} N_b(x_{r_n}, x_{r_n}, x_{s_n}, t) = 1 - \epsilon \qquad \text{--------------------}\ (5)$$

Again,

$$\begin{aligned} & N_b(x_{r_n}, x_{r_n}, x_{s_n-1}, t) \\ & \quad \geq N_b\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3k}\right) * N_b\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3k}\right) * N_b\left(x_{s_n-1}, x_{s_n-1}, x_{s_n}, \frac{t}{3k}\right) \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} N_b(x_{r_n}, x_{r_n}, x_{s_n-1}, t) > 1 - \epsilon \qquad \text{----------------------------------}\ (6)$$

Taking x = x_{r_n-1} and y = x_{s_n-1} in (1), we have

$$
\begin{aligned}
&\emptyset\left(N_b\left(x_{r_n}, x_{r_n}, x_{s_n}, t\right)\right)\\
&\leq b_1(x,y)\theta\left(\min\left\{\emptyset\left(N_b\left(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t\right)\right), \emptyset\left(N_b\left(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t\right)\right)\right\}\right)\\
&+ b_2(x,y)\psi\left(\emptyset\left(N_b\left(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t\right)\right) . \emptyset\left(N_b\left(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t\right)\right)\right)\\
&+ b_3(x,y)\emptyset\left(N_b\left(x_{r_n-1}, x_{r_n-1}, x_{s_n-1}, t\right)\right)\\
&+ b_4(x,y)\left[\emptyset\left(N_b\left(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t\right)\right) + \emptyset\left(N_b\left(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t\right)\right)\right]\\
&+\ b_5(x,y)\left[\emptyset\left(N_b\left(x_{r_n-1}, x_{r_n-1}, x_{s_n}, t\right)\right) + \emptyset\left(N_b\left(x_{r_n}, x_{r_n}, x_{s_n-1}, t\right)\right)\right]
\end{aligned}
$$

Taking $n \to \infty$ and by (4), (5), (6) and using the fact that p is asymptotically regular at x_0 we have ,

$$\emptyset(1-\epsilon) \leq b_3(x,y)\emptyset(1-\epsilon) + 2b_5(x,y)\emptyset(1-\epsilon) < \emptyset(1-\epsilon)$$

Which is a contradiction.

Thus $\{x_n\}$ is a Cauchy sequence. Since (X, N_b, *,k) is a complete N_b-fuzzy metric space, $\exists\, z \in X$ such that $x_n \to z$.

Existence of fixed point

$$
\begin{aligned}
&\emptyset\left(N_b(p(x_n), (p(x_n), p(z), t)\right)\\
&\qquad\leq\ b_1(x,y)\theta\left(\min\{\emptyset(N_b(x_n, x_n, x_{n+1}, t)), \emptyset(N_b(z, z, p(z), t))\}\right)\\
&\qquad+ b_2(x,y)\psi\left(\emptyset(N_b(x_n, x_n, x_{n+1}, t)), \emptyset(N_b(z, z, p(z), t))\right)\\
&\qquad+ b_3(x,y)\emptyset(N_b(x_n, x_n, z, t))\\
&\qquad+ b_4(x,y)\left[\emptyset(N_b(x_n, x_n, x_{n+1}, t)) + \emptyset(N_b(z, z, p(z), t))\right]\\
&\qquad+\ b_5(x,y)\left[\emptyset(N_b(x_n, x_n, p(z), t)) + \emptyset(N_b(z, z, x_{n+1}, t))\right]
\end{aligned}
$$

For $n \to \infty$

$$\lim_{n\to\infty} \emptyset\left(N_b(z, z, p(z), t)\right) \leq [\,b_4(x,y) + b_5(x,y)] \lim_{n\to\infty} \emptyset(N_b(z, z, p(z), t)\,)$$

$$\Rightarrow [1 - b_4(x,y) - b_5(x,y)] \lim_{n\to\infty} \emptyset\left(N_b(z, z, p(z), t)\right) \leq 0$$

$$\Rightarrow \lim_{n\to\infty} \emptyset\left(N_b(z, z, p(z), t)\right) = 0 \qquad (\text{ Since }\ 0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1)$$

$$\Rightarrow p(z) = z.$$

Uniqueness

If u is another fixed point of p in X , then

$$\begin{aligned}\emptyset(N_b,p(u),p(u),p(z),t)) &\leq b_1(x,y)\theta(\min\{\emptyset(N_b(u,u,p(u),t)),\emptyset(N_b(z,z,p(z),t))\}) \\ &+ b_2(x,y)\psi(\emptyset(N_b(u,u,p(u),t)),\emptyset(N_b(z,z,p(z),t))) \\ &+ b_3(x,y)(\emptyset(N_b(u,u,z,t)) \\ &+ b_4(x,y)[\emptyset(N_b(u,u,p(u),t)) + \emptyset(N_b(z,z,p(z),t))] \\ &+ b_5(x,y)\ [(\emptyset(N_b(u,u,p(z),t)) + \emptyset(N_b(z,z,p(u),t))]\end{aligned}$$

$$\Rightarrow \emptyset(N_b(u,u,z,t)) \leq b_3(x,y)\emptyset(N_b(u,u,z,t)) + 2b_5(x,y)\emptyset(N_b(u,u,z,t))$$

$$\Rightarrow [1 - b_3(x,y) - 2b_5(x,y)]\ \emptyset(N_b(u,u,z,t)) \leq 0$$

$\Rightarrow \emptyset(N_b(u,u,z,t)) = 0$ (Since $0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1$)

$$\Rightarrow u = z,$$

This completes the proof of theorem

COROLLARY 2.3 :-Let p, q : X → X be mapping on a complete fuzzy metric space (X, N_b, *, k) and ∅ be the altering distance function. Let p and q be asymptotically regular at a point $x_0 \in X$, and both satisfy the inequality (1). Moreover, if

$$\emptyset(N_b,(p(x),p(x),q(y),t)) \leq K\left(\emptyset(N_b,(x,x,y,t)) + \emptyset(N_b,(x,x,p(x),t)) + \emptyset(N_b,(y,y,q(y),t))\right), \text{ ----------------(7)}$$

Where $0 < K < 1 \ and \ x,y \in X,$

Then p and q have a unique common fixed point in X.

Proof. From theorem [1], both p and q have unique fixed points say, z and v respectively.

Since p and q satisfy (7),

$$\begin{aligned}\emptyset(N_b,(p(z),p(z),q(v),t)) \\ \leq\ K\left(\emptyset(N_b,(z,z,v,t)) + \emptyset(N_b,(z,z,p(z),t)) + \emptyset(N_b(v,v,q(v),t))\right),\end{aligned}$$

$$\Rightarrow (N_b,(z,z,v,t)) \leq K\left(\emptyset(N_b,(z,z,v,t)) + \emptyset(N_b,(z,z,z,t)) + \emptyset(N_b,(v,v,v,t))\right)$$

$$\Rightarrow (1-K)\emptyset(N_b,(z,z,v,t)) \leq 0$$

$$\Rightarrow \emptyset(N_b,(z,z,v,t)) \leq 0 (since\ K < 1)$$

$\Rightarrow z = v$

i.e.,p and q have a unique common fixed point.

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