

Certain Stationary Point Outcomes and Topology for Bicomplex-Valued N -Fuzzy Metric Space

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Abstract: This study extends common f -point results from their original establishment in bicomplex valued N -fuzzy metric spaces and presents bicomplex valued N -fuzzy metric spaces (BCVFNMs) and their topology. These results are further illustrated using a refined method that focuses on using more forgiving contractive requirements in the context of the bicomplex valued N -fuzzy metric. Taking into account a variety of other pertinent findings and expanding on the foundation established by earlier scholars. The theoretical underpinnings of fixed point theorems in bicomplex valued N -fuzzy metric spaces are strengthened by our new findings, which not only validate the established results but also significantly deepen and expand their applicability.

Keywords: N -metric spaces, Bicomplex valued N -fuzzy metric, common fixed point, t -norms.

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Introduction

Lotfi A. Zadeh [29] established the groundwork for fuzzy mathematics in 1965. The concept of fuzzy metric space was introduced by Kromosil and Michalek [13] in 1975. In contrast to the method proposed by [13], Ramotet. al. [13] used fuzzy metric space membership functions to find the fuzzy metric between two fuzzy sets. In 1994, Veeramani and George [6] introduced the updated idea of Continuous t -norm, which redefined fuzzy metric space. In his revolutionary work, Segre [24] established the foundation for the study of special algebras by presenting a novel viewpoint in the form of commutative generalizations beyond complex numbers, including tricomplex and bicomplex numbers. Price [17] expanded the topic by creating bicomplex algebra and function theory, building on Segre's work. Letter on, numerous authors and researchers generalized various type of metric spaces like fuzzy 2-metric space, D -metric space, G -metric space-metric space, $Q(G)$ -metric space, D^* -metric space M -metric spaces-metric space and have thoroughly and extensively examined a variety of issues pertaining to this space from a variety of perspectives. In 2015, N. Malviya [21] introduced the new notion of N -fuzzy metric space, Pseudo N -fuzzy metric space and describes some of their properties and examples. Fuzzy complex numbers and fuzzy complex analysis were initially introduced by Buckley [2,3]. Buckley's study motivated some researchers to continue their

research on fuzzy complex numbers. In this series, Ramot et al. [18, 19] generalized fuzzy sets to complex fuzzy sets. According to Ramot, a membership function that extends beyond $[0, 1]$ to the unit circle in the complex plane defines the complex fuzzy set. Azam et al. [1] presented the concept of complex valued metric space and gave enough conditions for the presence of a common f -point of a pair of mappings that meet contractive constraints. The concept of complex valued fuzzy metric spaces was later presented by D. Singh and Kumam [25] using the complex valued continuous t -norm and associated topologies. After looking at a number of fixed point's outcomes, I. Demir [5] introduced the idea of complex valued fuzzy b -metric spaces. Bicomplex valued fuzzy metric space and bicomplex valued fuzzy b -metric space were introduced by RakeshTiwari et al. [27] in 2022 along with various common f -point results. Inspired and motivated by the work of RakeshTiwari et al. [27], we employ sometimes weakly compatible mappings in this study to establish certain common f -point facts.

In this study, we apply the E.A.- and CLR-properties, which were developed by earlier researchers, to the setting of bicomplex valued N -fuzzy metric spaces ($BCVFNMs$). The findings of the common fixed point theorem are generalized and expanded upon by our results, which also greatly enhance the findings of current research.

2. Preliminaries and Main Results :

Definition 2.1[25]Definition 2.1: A map $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is called continuous triangular norm, if it's satisfied the following circumstance for $a, b, c, d \in [0,1]$:

- (i) $a * b = b * a$ (*Symmetry*);
- (ii) $a * (b * c) = (a * b) * c$ (*Associativity*);
- (iii) $a * 1 = a$ (*boundary condition*);
- (iv) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ (*Monotonicity*);

Example 2.2:Let (X, d) be a metric space,

- (a) $a * b = \min\{a, b\}$ (*minimum t – norm*)

for all $a, b \in [0,1]$, Define as $\mathcal{M}(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$;

- (b) $a * b = ab$ (*Product t – norm*)

Let $X = [0, \infty)$, $a * b = ab$ for every, $a, b \in [0,1]$ and d is usual metric. Define a function

$$\mathcal{M}(x, y, t) = e^{-\frac{d(x,y)}{t}} ; x, y, t \in X, t > 0 \text{ then } (X, \mathcal{M}, *) \text{ is a FMS.}$$

Definition 2.3[6]:If X is an arbitrary set, $*$ is a continuous t_{norm} on $[0, \infty]$ and M is a fuzzy set on $\mathfrak{X}^3 \times (0, \infty)$ that all satisfies the following criteria, then the 3-triplet $(\mathfrak{X}, M, *)$ is known as fuzzy metric space (FMS):

- (1) $M(x, y, t) > 0$;
- (2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

$$(3) M(x, y, t) = M(y, x, t);$$

$$(4) M(x, y, t_1) * M(x, y, t_2) \leq M(x, z, t_1 + t_2);$$

$$\forall x, y, z \in \mathfrak{X} \text{ and } t_1, t_2 > 0$$

$$(5) M(x, y, .): (0, \infty) \rightarrow (0, 1] \text{ is continuous function;}$$

Here $(\mathfrak{X}, M, *)$ is taken into consideration as the degree to which x and y are close to t . Since a classical metric space may only accept the value 0 and not the value ∞ , Property (1) is warranted. $M(x, x, t) = 1$ for all $x \in \mathfrak{X}$ and $t > 0$ and $M(x, y, t) < 1$ for all $x \neq y$ and $t > 0$ are equivalents to property (2). Property (2) states that if $x=y$, then $M(x, x, t) = 1$, indicating that the degree of nearness is optimal.

Definition 2.4[15]: If X is an arbitrary set, $*$ is a continuous t_{norm} on $[0, \infty]$ and M is a fuzzy set on $\mathfrak{X}^3 \times (0, \infty)$ that all satisfies the following criteria, and then the 3-triplet $(\mathfrak{X}, M, *)$ is known as N -fuzzy metric space ($N - FMS$):

$$(1) N(x, y, z, t) > 0;$$

$$(2) N(x, y, z, t) = 1 \text{ for all } t > 0 \text{ iff } x = y = z;$$

$$(3) N(x, y, z, t_1 + t_2 + t_3) \geq N(x, x, a, t_1) * N(y, y, a, t_2) * N(z, z, a, t_3);$$

$$\forall x, y, z \in \mathfrak{X} \text{ and } t_1, t_2, t_3 > 0$$

$$(4) N(x, y, z, .): (0, \infty) \rightarrow [0, 1] \text{ is continuous;}$$

In his discussion of bicomplex numbers, Segre [22] defined the set of bicomplex numbers as follows:

$$\mathfrak{E}_2 = \{w: w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathfrak{E}_0\}$$

$$\mathfrak{E}_2 = \{w: w = z_1 + z_2 i_2, z_1, z_2 \in \mathfrak{E}_1\}$$

The symbol, $\mathfrak{E}_0, \mathfrak{E}_1$, and \mathfrak{E}_2 as a set of real, complex and bicomplex numbers respectively.

Where $z_1 = a_1 + a_2 i_1$, $z_2 = a_3 + a_4 i_1$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$ and $i_1 i_2 = i_2 i_1$. the product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. the products of all units are commutative and satisfy:

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

For a bicomplex number $w = z_1 + z_2 i_2$, the norm is denoted by $\|z_1 + z_2 i_2\|$ and defined by

$$(|z_1|^2 + |z_2|^2)^{1/2} = (|z_1 - z_2 i_1|^2 + |z_1 + z_2 i_1|^2)^{1/2}.$$

If we take $w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$ then the norm of w is defined by

$$\|w\| = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{1/2}$$

Choi et al. [4] formulated the partial order relation \lesssim_{i_2} on \mathbb{C}_2 as follows $\tau \lesssim_{i_2} v$ if and only if $z_1 \leq w_1$ and $z_2 \leq w_2$, for $\tau = z_1 + z_2 i_2$ and $v = w_1 + w_2 i_2$ are two bicomplex numbers. It follows that $\tau \lesssim_{i_2} v$. Therefore, if one of the following requirements is $\tau \lesssim_{i_2} v$.

(i) $z_1 = w_1, z_2 = w_2$,

(ii) $z_1 \prec w_1, z_2 = w_2$,

(iii) $z_1 = w_1, z_2 \prec w_2$,

(iv) $z_1 \prec w_1, z_2 \prec w_2$,

We write $\tau \approx_{i_2} v$ if $\tau \lesssim_{i_2} v$ and $t \neq v$ and one of (ii), (iii) and (iv) are satisfied and we will write $\tau \prec v$ if only (iv) is satisfied.

Fuzzy version was defended by D. Singh et al. [27] as the complex valued continuous t-norm and complex valued fuzzy metric space:

Definition 2.4[27]: A binary operation $*$: $r_s e^{i\theta} \times r_s e^{i\theta} \rightarrow r_s e^{i\theta}$, where $r_s \in [0,1], \theta \in [0, \frac{\pi}{2}]$ is called complex valued triangular norm (or t-norm) if it satisfies the following properties:

(a) $*$ is associative and commutative.

(b) $*$ is continuous.

(c) $\alpha * e^{i\theta} = \alpha$, for all $\alpha \in e^{i\theta}, \theta \in [0, \frac{\pi}{2}]$.

(d) $a * b \lesssim_{i_2} c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in r_s e^{i\theta}, r_s \in [0,1]$.

Definition 2.5[27]: Fuzzy metric with complex values the triplet $(X, \mathcal{M}, *)$ is space, Where $\mathcal{M}: X^2 \times (0, \infty) \rightarrow r_s e^{i\theta}$ and X is non-empty set, $*$ is a complex valued continuous, and t-norm. The fuzzy set $\rightarrow r_s e^{i\theta}$ has complex values and satisfies the following characteristics:

($M_{CFS} - 1$) $\mathcal{M}(x, y, t) \succ_{i_2} 0$;

($M_{CFS} - 2$) $\mathcal{M}(x, y, t) = e^{i\theta}$, for all $t > 0$ iff $x = y$;

($M_{CFS} - 3$) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;

($M_{CFS} - 4$) $\mathcal{M}(x, y, p) * \mathcal{M}(y, z, q) \succeq_i \mathcal{M}(x, z, p + q)$;

($M_{CFS} - 5$) $\mathcal{M}(x, y, \cdot): (0, \infty) \rightarrow e^{i\theta}$ is continuous;

For all $x, y, z \in X, p, q > 0, r_s \in [0,1], \theta \in [0, \frac{\pi}{2}]$.

Then $(X, \mathcal{M}, *)$ is called complex valued fuzzy metric space.

Example 2.6: let (X, d) be a metric space, define $u * v = \min\{u, v\}$ (or $u * v = uv$).

For each $t > 0, x, y \in X$, define $\mathcal{M}(x, y, t) = e^{i\theta} \frac{ht^n}{ht^n + md(x,y)}$, $h, m, n \in N$.

If $h = m = n = 1$ then $\mathcal{M}(x, y, t) = e^{i\theta} \frac{t}{t+d(x,y)}$

Example 2.7: (X, d) be a metric space, define $u * v = \max\{u + v - e^{i\theta}, 0\}$, for a fix $\theta \in [0, \frac{\pi}{2}]$.

The term "standard complex valued fuzzy metric space" was used to describe this metric-induced complex valued fuzzy metric space. This paper aims to verify various fixed point results and create the concept of bicomplex valued N-fuzzy metric spaces using t-norms. We provide instances to support our findings.

Reader can refer references [1-29] to study of related definitions

Definition 2.8[27]: A binary operation $*$: $r_s(1 + i_2)e^{i_1\theta} \times r_s(1 + i_2)e^{i_1\theta} \rightarrow r_s(1 + i_2)e^{i_1\theta}$, where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ is called bicomplex valued triangular norm (or t-norm) if it satisfies the following properties:

(a) $*$ is associative and commutative.

(b) $*$ is continuous.

$$(c) \alpha * (1 + i_2)e^{i_1\theta} = \alpha, \text{ for all } \alpha \in (1 + i_2)e^{i_1\theta}.$$

(d) $a * b \lesssim_{i_2} c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in r_s(1 + i_2)e^{i_1\theta}$, $r_s \in [0,1]$.

Definition 2.9: A bicomplex valued \mathcal{N} –fuzzy metric space is a triplet $(X, \mathcal{N}, *)$ Where X is non-empty set, $*$ is a bicomplex valued continuous is t-norm and $\mathcal{N}: X^3 \times (0, \infty) \rightarrow r_s(1 + i_2)e^{i_1\theta}$ is bicomplex valued fuzzy set, satisfying the following properties:

$$(\mathcal{N}_{BCFS} - 1) \mathcal{N}(x, y, z, t) \succ_{i_2} 0;$$

$$(\mathcal{N}_{BCFS} - 2) \mathcal{N}(x, y, z, t) = (1 + i_2)e^{i_1\theta}, \text{ for all } t > 0 \text{ iff } x = y = z;$$

$$(\mathcal{N}_{BCFS} - 3) \mathcal{N}(x, y, z, t_1 + t_2 + t_3) \geq \mathcal{N}(x, x, a, t_1) * \mathcal{N}(y, y, a, t_2) * \mathcal{N}(z, z, a, t_3);$$

$$\forall x, y, z \in \mathfrak{X} \text{ and } t_1, t_2, t_3 > 0;$$

$$(\mathcal{N}_{BCFS} - 4) \mathcal{N}(x, y, z, \cdot): (0, \infty) \rightarrow r_s(1 + i_2)e^{i_1\theta} \text{ is continuous};$$

$$\text{For all } x, y, z \in X, p, q > 0, r_s \in [0,1], \theta \in [0, \frac{\pi}{2}].$$

Then $(X, \mathcal{N}, *)$ is called bicomplex valued \mathcal{N} – fuzzy metric spaces .

Example 2.10: let $X = R$, and S be a S-metric on X define by define

$$S(x, y, z) = |x - z| + |y - z| \text{ or } |y + z - 2x| + |y - z|.$$

Define $a * b * c = abc$, for all $a, b, c \in r_s(1 + i_2)e^{i_1\theta}$,

Where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ and \mathcal{N} be the function on $\mathcal{N}: X^3 \times (0, \infty) \rightarrow r_s(1 + i_2)e^{i_1\theta}$

defined by $\mathcal{N}(x, y, z, t) = r_s(1 + i_2)e^{i_1\theta} - \frac{t}{t + S(x, y, z)}$, for all $x, y, z \in X$ and $t \in$

$(0, \infty)$ then $(X, \mathcal{N}, *)$ is a bicomplex valued \mathcal{N} – fuzzy metric spaces.

Lemma 2.11: Let $(X, \mathcal{N}, *)$ BCNFM space with $\lim_{t \rightarrow \infty} \mathcal{N}(x, x, y, t) = (1 + i_2)e^{i_1\theta}$, $\forall x, y \in X, t \in (0,1)$ then $\mathcal{N}(x, x, y, kt) \geq \mathcal{N}(x, x, y, t) \Rightarrow x = y, k \in (0,1)$.

Topology of Bicomplex Valued \mathcal{N} - Fuzzy Metric Spaces.

Example 2.12: Let $(X, \mathcal{N}, *)$ be a bicomplex valued \mathcal{N} -fuzzy metric space with $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in \mathbb{C}_2$, $0 < r < (1 + i_2)e^{i_1\theta}$ is defined by $B(x, r, t) = \{y \in X : N(y, y, x, t) > (1 + i_2)e^{i_1\theta} - r\}$. The collection $\{B(x, r, t) : x \in X, 0 < r < (1 + i_2)e^{i_1\theta}, t > 0\}$, where $\theta \in [0, \frac{\pi}{2}]$, is a neighborhood system for topology τ on X induced by the bicomplex valued \mathcal{N} -fuzzy metric space.

Definition 2.13: Let $(X, \mathcal{N}, *)$ be a bicomplex valued \mathcal{N} -fuzzy metric space. Then, $(X, \mathcal{N}, *)$ is called a Hausdorff space if for any two distinct points $u, v \in X$, there exist two open balls $B(u, r_1, t_1) = B(u, (1 + i_2)e^{i_1\theta} - r_1, t_1)$ and $B(v, r_2, t_2) = B(v, (1 + i_2)e^{i_1\theta} - r_2, t_2)$ Such that $B(u, r_1, t_1) \cap B(v, r_2, t_2) = \phi$

Definition 2.14: Let $(X, \mathcal{N}, *)$ be a bicomplex valued \mathcal{N} -fuzzy metric space. A subset A of X is said to be bounded if and only if there exist $t > 0$ and $r \in \mathbb{C}_2$, $0 < r < (1 + i_2)e^{i_1\theta}$ Such that $N(y, y, x, t) > (1 + i_2)e^{i_1\theta} - r$, for all $x, y \in A$.

Definition 2.15: A sequence $\{x_n\}$ in a bicomplex valued \mathcal{N} -fuzzy metric space $(X, \mathcal{N}, *)$ is a Cauchy sequence if and only if

$$\lim_{n \rightarrow \infty} N(x_{n+p}, x_{n+p}, x_n, t) = (1 + i_2)e^{i_1\theta}, p > 0, t > 0$$

$$\text{or } \lim_{n \rightarrow \infty} |N(x_{n+p}, x_{n+p}, x_n, t)| = 1, p > 0, t > 0.$$

Definition 2.16: A sequence $\{x_n\}$ in a bicomplex valued \mathcal{N} -fuzzy metric space $(X, \mathcal{N}, *)$ is convergent to $x_n \rightarrow x$ if and only if

$$\lim_{n \rightarrow \infty} N(x_n, x_n, x, t) = (1 + i_2)e^{i_1\theta} \text{ as } n \rightarrow \infty$$

$$\text{or } |N(x_n, x_n, x, t)| \rightarrow 1.$$

Proposition 2.17: Let $(X, \mathcal{N}, *)$ be a bicomplex valued \mathcal{N} -fuzzy metric space, then for all $x, y \in X$ and $t > 0$, then $N(x, x, y, t) = N(y, y, x, t)$.

Proof:- Since bicomplex valued \mathcal{N} -fuzzy metric space is induced by S -metric and in S -metric space $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$. Therefore in bicomplex valued \mathcal{N} -fuzzy metric space. $N(x, x, y, t) = N(y, y, x, t)$ for all $x, y \in X$ and $t > 0$.

Proposition 2.18: Every open ball is an open set in bicomplex valued \mathcal{N} -fuzzy metric space.

Proof: Consider an open ball $B(x, r, t)$. To show $B(x, r, t)$ to be open we show that at every point of $B(x, r, t)$, there exists an open ball contained in $B(x, r, t)$.

Let $y \in B(x, r, t) \Rightarrow N(x, x, y, t) > (1 + i_2)e^{i_1\theta} - r$, where $r \in \mathbb{C}_2$ and $0 < r < (1 + i_2)e^{i_1\theta}$, thus we can find a $t_0, 0 < t_0 < t$ such that $N(x, x, y, t_0) > (1 + i_2)e^{i_1\theta} - r$.

Let $r_0 = N(x, x, y, t_0) > (1 + i_2)e^{i_1\theta} - r$. Then we can find s , where $0 < u < (1 + i_2)e^{i_1\theta}$, such that $r_0 > (1 + i_2)e^{i_1\theta} - u > (1 + i_2)e^{i_1\theta} - r$.

For given r_0 and u , where $r_0 > (1 + i_2)e^{i_1\theta} - u$, we can find $r_1, 0 < r_1 < (1 + i_2)e^{i_1\theta}$ such that $r_0 * r_1 > (1 + i_2)e^{i_1\theta} - u$.

Consider the ball $B(y, (1 + i_2)e^{i_1\theta} - r_1, t - t_0)$.

We assert that $B(y, (1 + i_2)e^{i_1\theta} - r_1, t - t_0) \subset B(x, r, t)$.

$$\begin{aligned} \text{Let } z \in B(y, (1 + i_2)e^{i_1\theta} - r_1, t - t_0) &\Rightarrow N(y, y, z, t - t_0) \\ &> ((1 + i_2)e^{i_1\theta} - (1 + i_2)e^{i_1\theta} - r_1) = r_1. \end{aligned}$$

Since $N(y, y, z, t - t_0) > r_1$.

$$\begin{aligned} \text{Consider } N(x, x, z, t) &\geq n(x, x, y, t_0) * N(y, y, z, t - t_0) \\ &\geq r_0 * r_1 \\ &\geq (1 + i_2)e^{i_1\theta} - u \geq (1 + i_2)e^{i_1\theta} - r. \end{aligned}$$

$$N(x, x, z, t) = ((1 + i_2)e^{i_1\theta} - r).$$

This amounts to say that $z \in B(x, r, t)$.

Hence $B(y, (1 + i_2)e^{i_1\theta} - r_1, t - t_0) \in B(x, r, t)$.

This shows that $B(x, r, t)$ is an open set.

Proposition 2.19: Every bicomplex valued \mathcal{N} -fuzzy metric space is Hausdorff.

Proof:- Let $(X, \mathcal{N}, *)$ be bicomplex valued \mathcal{N} -fuzzy metric space. Let x, y be two distinct points of X then $0 < N(x, x, y, t) < (1 + i_2)e^{i_1\theta}$. Let $N(x, x, y, t) = r$ for some $r \in \mathbb{E}_2$ then $0 < r < (1 + i_2)e^{i_1\theta}$. For each $r_0 \in \mathbb{E}_2$ then $r < r_0 < (1 + i_2)e^{i_1\theta}$, we can find an r_1 such that $r_1 * r_1 * r_1 \geq r_0$.

Now consider the open balls $B(x, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3})$ and $B(y, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3})$.

$$\text{Clearly } B(x, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3}) \cap B(y, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3}) = \phi$$

For if there exists $z \in B(x, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3}) \cap B(y, (1 + i_2)e^{i_1\theta} - r_1, \frac{t}{3})$

Then

$$\begin{aligned} r &= N(x, x, y, t) \\ &\geq N(x, x, z, \frac{t}{3}) * N(x, x, z, \frac{t}{3}) * N(y, y, z, \frac{t}{3}) \\ &> r_1 * r_1 * r_1 \\ &\geq r_0 > r. \end{aligned}$$

This is a contradiction. Therefore $(X, \mathcal{N}, *)$ is Hausdorff.

Lemma 2.20:- $\mathcal{N}(x, x, y, \cdot)$ is non-decreasing for all x, y in X .

Proof:- Suppose that $\mathcal{N}(x, x, y, t) > \mathcal{N}(x, x, y, s)$ for some $0 < t < s$.

$$\begin{aligned} \text{Then } \mathcal{N}(x, x, x, \frac{s}{2}) * \mathcal{N}(x, x, x, \frac{s}{2} - t) * \mathcal{N}(y, y, x, t) &\leq \mathcal{N}(x, x, y, s) \\ &< \mathcal{N}(x, x, y, t) \end{aligned}$$

By definition 3.1 , we have $\mathcal{N}(x, x, x, \frac{s}{2}) = (1 + i_2)e^{i_1\theta}$

and $\mathcal{N}(x, x, x, \frac{s}{2} - t) = (1 + i_2)e^{i_1\theta}$ Thus

$$\mathcal{N}(y, y, x, t) \leq \mathcal{N}(x, x, y, s)$$

$\mathcal{N}(x, x, y, t) \leq \mathcal{N}(x, x, y, s)$ [As $\mathcal{N}(x, x, y, t) = \mathcal{N}(y, y, x, t)$] a contradiction.

Application in Fixed Point Theory

As an application of weak compatible maps and the property(EA), CLR-Property

Theorem 3.1: Let Σ, Y, ϑ and Ω be self maps of $BCVNFMs$ space $(X, \mathcal{N}, *)$ with $\lim_{t \rightarrow \infty} \mathcal{N}(x, x, y, t) = (1 + i_2)e^{i_1\theta}, \forall x, y \in X, t \in (0, 1)$ satisfying the following conditions:

- (a) $\Sigma(X) \subseteq \Omega(X), Y(X) \subseteq \vartheta(X)$;
- (b) (Σ, ϑ) and (Y, Ω) are weakly compatible pairs;
- (c) (Σ, ϑ) or (Y, Ω) Satisfies the (E. A.) – property;
- (d) For some $\varphi \in \emptyset$, there exist $k \in (0, 1)$ such that for all $x, y, \in X, t > 0$

$$\mathcal{N}(\Sigma x, \Sigma x, Yy, kt) \gtrsim \text{Min}\{\mathcal{N}(\vartheta x, \vartheta x, \Omega y, t), \mathcal{N}(\Sigma x, \Sigma x, \vartheta x, t), \mathcal{N}(Yy, Yy, \Omega y, t)\}$$

If the range of one of the maps Σ, Y, ϑ and Ω is a complete subspace of X , then Σ, Y, ϑ and Ω have a unique common fixed point in X .

Proof:- If the pair (Y, Ω) satisfies the property (E. A.) then \exists a sequence $\{x_n\}$ such that $Yx_n \rightarrow z$ and $\Omega x_n \rightarrow z$ for some $z \in X$ as $n \rightarrow \infty$.

Since $Y(X) \subseteq \vartheta(X), \exists$ a sequence $\{y_n\}$ such that $Yx_n = \vartheta y_n$. Hence $\vartheta y_n \rightarrow z$ as $n \rightarrow \infty$.

Now we claim $\Sigma y_n \rightarrow z$ as $n \rightarrow \infty$.

Suppose $\Sigma y_n \rightarrow p (\neq z) \in X$, then by inequality (d),

$$\begin{aligned} \mathcal{N}(\Sigma y_n, \Sigma y_n, Yx_n, kt) &\gtrsim \text{Min}\{\mathcal{N}(\vartheta y_n, \vartheta y_n, \Omega x_n, t), \mathcal{N}(\Sigma y_n, \Sigma y_n, \vartheta y_n, t), \mathcal{N}(Yx_n, Yx_n, \Omega x_n, t)\} \\ \mathcal{N}(\Sigma y_n, \Sigma y_n, Yx_n, kt) &\gtrsim \text{Min}\{\mathcal{N}(Yx_n, Yx_n, \Omega x_n, t), \mathcal{N}(\Sigma y_n, \Sigma y_n, Yx_n, t), \mathcal{N}(Yx_n, Yx_n, \Omega x_n, t)\} \end{aligned}$$

Taking limit as $n \rightarrow \infty$

$$\mathcal{N}(p, p, z, kt) \gtrsim \text{Min}\{\mathcal{N}(z, z, z, t), \mathcal{N}(p, p, z, t), \mathcal{N}(z, z, z, t)\}$$

$$N(p, p, z, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, N(p, p, z, t), (1 + i_2)e^{i_1\theta}\}$$

$N(p, p, z, kt) \gtrsim N(p, p, z, t) \Rightarrow p = z$. This shows that $\Sigma y_n \rightarrow z$ as $n \rightarrow \infty$.

Suppose that $\vartheta(X)$ is a complete subspace of X . Then $z = \vartheta u$ for some $u \in X$.

Subsequently, we have $\Sigma y_n \rightarrow \vartheta u, Yx_n \rightarrow \vartheta u, \Omega x_n \rightarrow \vartheta u$ and $\vartheta y_n \rightarrow \vartheta u$ as $n \rightarrow \infty$.

by inequality (d),

$$N(\Sigma u, \Sigma u, \vartheta u, kt) \gtrsim \text{Min}\{N(\vartheta u, \vartheta u, \vartheta u, t), N(\Sigma u, \Sigma u, \vartheta u, t), N(\vartheta u, \vartheta u, \vartheta u, t)\}$$

$$N(\Sigma u, \Sigma u, \vartheta u, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, N(\Sigma u, \Sigma u, \vartheta u, t), (1 + i_2)e^{i_1\theta}\}$$

$$N(\Sigma u, \Sigma u, \vartheta u, kt) \gtrsim N(\Sigma u, \Sigma u, \vartheta u, t) \Rightarrow \Sigma u = \vartheta u.$$

Since the pair (Σ, ϑ) is weakly compatible i. e. $\Sigma \vartheta u = \vartheta \Sigma u$ implies that $\Sigma \vartheta u = \vartheta \Sigma u$ and then $\Sigma \Sigma u = \Sigma \vartheta u = \vartheta \Sigma u = \vartheta \vartheta u$.

On the other hand, since $\Sigma(X) \subseteq \Omega(X), \exists v \in X$ such that $\Sigma u = \Omega v$.

We show that $\Omega v = Yv$.

$$N(\Sigma u, \Sigma u, Yv, kt) \gtrsim \text{Min}\{N(\vartheta u, \vartheta u, \Omega v, t), N(\Sigma u, \Sigma u, \vartheta u, t), N(Yv, Yv, \Omega v, t)\}$$

$$N(\Omega v, \Omega v, Yv, kt) \gtrsim \text{Min}\{N(\vartheta u, \vartheta u, \Omega v, t), N(\Omega v, \Omega v, \vartheta u, t), N(Yv, Yv, \Omega v, t)\}$$

$$N(\Omega v, \Omega v, Yv, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}, N(Yv, Yv, \Omega v, t)\}$$

Since $N(x, x, y, t) = N(y, y, x, t)$

$$N(\Omega v, \Omega v, Yv, kt) \gtrsim N(\Omega v, \Omega v, Yv, t)$$

Thus $\Omega v = Yv$.

This implies that $\Sigma u = \vartheta u = \Omega v = Yv$.

Now (Y, Ω) is weakly compatible pairs then $Y\Omega v = \Omega Yv$ and then $\Omega \Omega v = Y\Omega v = \Omega Yv = Y Yv$.

Now we show that Σu is common fixed point of Σ, Y, ϑ and Ω .

From inequality (d)

$$N(\Sigma \Sigma u, \Sigma \Sigma u, Yv, kt) \gtrsim \text{Min}\{N(\vartheta \Sigma u, \vartheta \Sigma u, \Omega v, t), N(\Sigma \Sigma u, \Sigma \Sigma u, \vartheta \Sigma u, t), N(Yv, Yv, \Omega v, t)\}$$

$$N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma u, kt) \gtrsim \text{Min}\{N(\Sigma \Sigma u, \Sigma \Sigma u, \Omega v, t), N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma \Sigma u, t), N(Yv, Yv, Yv, t)\}$$

$$N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma u, kt) \gtrsim \text{Min}\{N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma u, t), (1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}\}$$

$$N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma u, kt) \gtrsim N(\Sigma \Sigma u, \Sigma \Sigma u, \Sigma u, t) \Rightarrow \Sigma \Sigma u = \Sigma u$$

Therefore $\Sigma \Sigma u = \Sigma u = \vartheta \Sigma u$ so that Σu is common fixed point of Σ and ϑ . similarly we can prove that Yv is common fixed point of Y and Ω .

Since we have $\Sigma u = Yv$. So that Σu is common fixed point of Σ, Y, ϑ and Ω . The proof is similar when $\Omega(X)$ is assumed to be a complete subspace of X . The cases in which $\Sigma(X)$ or $Y(X)$ is a complete subspace of X are similar to the cases in which $\Omega(X)$ or $\vartheta(X)$ respectively, is complete since $\Sigma(X) \subseteq \Omega(X), Y(X) \subseteq \vartheta(X)$.

Uniqueness: Suppose α' is another unique common fixed point.

If $\Sigma u = \Omega u = Y u = \vartheta u = u$ and $\Sigma \alpha' = \Omega \alpha' = Y \alpha' = \vartheta \alpha' = \alpha'$ then by inequality (d)

$$N(\Sigma u, \Sigma u, Y \alpha', kt) \gtrsim \text{Min}\{N(\vartheta u, \vartheta u, \Omega \alpha', t), N(\Sigma u, \Sigma u, \vartheta u, t), N(Y \alpha', Y \alpha', \Omega \alpha', t)\}$$

$$N(u, u, \alpha', kt) \gtrsim \text{Min}\{N(u, u, \alpha', t), N(u, u, u, t), N(\alpha', \alpha', \alpha', t)\}$$

$$N(u, u, \alpha', kt) \gtrsim \text{Min}\{N(u, u, \alpha', t), (1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}\}$$

$$N(u, u, \alpha', kt) \gtrsim N(u, u, \alpha', t) \Rightarrow u = \alpha'$$

Therefore, the common fixed point is unique.

Theorem 3.2: Let Σ, Y, ϑ and Ω be self maps of BCNFM space $(X, \mathcal{N}, *)$ with

$$\lim_{t \rightarrow \infty} N(x, x, y, t) = (1 + i_2)e^{i_1\theta}, \forall x, y \in X, t \in (0, 1) \text{ satisfying the following conditions:}$$

(a) $\Sigma(X) \subseteq \Omega(X), Y(X) \subseteq \vartheta(X)$;

(b) (Σ, ϑ) and (Y, Ω) are weakly compatible pairs;

(c) if either pair (Σ, ϑ) meets $CLR_\Sigma - Property$ or pair (Y, Ω) meets $CLR_Y - Property$;

(d) For some $\varphi \in \emptyset$, there exist $k \in (0, 1)$ such that for all $x, y \in X, t > 0$

$$N(\Sigma x, \Sigma x, Y y, kt) \gtrsim \text{Min}\{N(\vartheta x, \vartheta x, \Omega y, t), N(\Sigma x, \Sigma x, \vartheta x, t), N(Y y, Y y, \Omega y, t)\}$$

Then Σ, Y, ϑ and Ω have a unique common fixed point in X .

Proof: Since the pair (Y, Ω) meets $CLR_Y - Property$ then $\exists \{x_n\}$ in X

Such that $\lim_{n \rightarrow \infty} Y x_n = \lim_{n \rightarrow \infty} \Omega x_n = Y \varrho$, for some $\varrho \in X$. now $Y(X) \subseteq \vartheta(X)$ we have $Y \varrho = \vartheta \sigma$, for some $\sigma \in X$.

Now we claim $\Sigma \sigma = \vartheta \sigma (= \mu)$.

Suppose it is not true then by inequality (d)

$$N(\Sigma \sigma, \Sigma \sigma, Y \varrho, kt) \gtrsim \text{Min}\{N(\vartheta \sigma, \vartheta \sigma, Y \varrho, t), N(\Sigma \sigma, \Sigma \sigma, \vartheta \sigma, t), N(Y \varrho, Y \varrho, Y \varrho, t)\}$$

$$N(\Sigma \sigma, \Sigma \sigma, Y \varrho, kt) \gtrsim \text{Min}\{N(\vartheta \sigma, \vartheta \sigma, \vartheta \sigma, t), N(\Sigma \sigma, \Sigma \sigma, Y \varrho, t), N(Y \varrho, Y \varrho, Y \varrho, t)\}$$

$$N(\Sigma \sigma, \Sigma \sigma, Y \varrho, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, N(\Sigma \sigma, \Sigma \sigma, Y \varrho, t), (1 + i_2)e^{i_1\theta}\}$$

$$N(\Sigma \sigma, \Sigma \sigma, Y \varrho, kt) \gtrsim N(\Sigma \sigma, \Sigma \sigma, Y \varrho, t) \Rightarrow \Sigma \sigma = Y \varrho, \text{ which is contradiction.}$$

Thus $\Sigma \sigma = \vartheta \sigma (= \mu)$.

Hence $\Sigma\sigma = \vartheta\sigma = Y\varrho = \mu$.

Thus σ is coincidence point of (Σ, ϑ) .

Now pair (Σ, ϑ) is weakly compatible so that $\Sigma\vartheta\sigma = \vartheta\Sigma\sigma = \Sigma\mu = \vartheta\mu$.

As given $\Sigma(X) \subseteq \Omega(X)$ then $\exists, \partial \in X$ such that $\Sigma\sigma = \Omega\partial$

Now to prove $Y\partial = \mu$.

From inequality (d)

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{N(\vartheta\sigma, \vartheta\sigma, \Omega\partial, t), N(\Sigma\sigma, \Sigma\sigma, \vartheta\sigma, t), N(Y\partial, Y\partial, \Omega\partial, t)\}$$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{N(\mu, \mu, \Omega\partial, t), N(\mu, \mu, \mu, t), N(Y\partial, Y\partial, \Omega\partial, t)\}$$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{N(\mu, \mu, \Sigma\sigma, t), N(\mu, \mu, \mu, t), N(Y\partial, Y\partial, \Sigma\sigma, t)\}$$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{N(\mu, \mu, \mu, t), N(\mu, \mu, \mu, t), N(Y\partial, Y\partial, \Sigma\sigma, t)\}$$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}, N(Y\partial, Y\partial, \Sigma\sigma, t)\}$$

Since $N(x, x, y, t) = N(y, y, x, t)$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim \text{Min}\{(1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}, N(\Sigma\sigma, \Sigma\sigma, Y\partial, t)\}$$

$$N(\Sigma\sigma, \Sigma\sigma, Y\partial, kt) \gtrsim N(\Sigma\sigma, \Sigma\sigma, Y\partial, t)$$

This gives $\Sigma\sigma = Y\partial = \mu$.

Hence $\Sigma\sigma = \vartheta\sigma = Y\partial = \Omega\partial = \mu$.

Which shows that ∂ is coincidence point of (Y, Ω) .

Since the pair (Y, Ω) are weakly compatible i.e. $Y\Omega\partial = \Omega Y\partial \Rightarrow Y\mu = \Omega\mu$.

Thus μ is common coincidence point of (Σ, ϑ) . and (Y, Ω) .

Uniqueness: Suppose α' is another unique common fixed point.

If $\Sigma\mu = \Omega\mu = Y\mu = \vartheta\mu = \mu$ and $\Sigma\alpha' = \Omega\alpha' = Y\alpha' = \vartheta\alpha' = \alpha'$ then by inequality (d)

$$N(\Sigma\mu, \Sigma\mu, Y\alpha', kt) \gtrsim \text{Min}\{N(\vartheta\mu, \vartheta\mu, \Omega\alpha', t), N(\Sigma\mu, \Sigma\mu, \vartheta\mu, t), N(Y\alpha', Y\alpha', \Omega\alpha', t)\}$$

$$N(\mu, \mu, \alpha', kt) \gtrsim \text{Min}\{N(\mu, \mu, \alpha', t), N(\mu, \mu, \mu, t), N(\alpha', \alpha', \alpha', t)\}$$

$$N(\mu, \mu, \alpha', kt) \gtrsim \text{Min}\{N(\mu, \mu, \alpha', t), (1 + i_2)e^{i_1\theta}, (1 + i_2)e^{i_1\theta}\}$$

$$N(\mu, \mu, \alpha', kt) \gtrsim N(\mu, \mu, \alpha', t) \Rightarrow \mu = \alpha'$$

Therefore, the common fixed point is unique.

Example 3.3: Let $X = [0,20]$ and S be an S -metric on X defined by $S : X^3 \times [0, \infty)$ such that $S(x, y, z) = |x - z| + |y - z|$. Clearly $(X, \mathcal{N}, *)$ is $BCVFM S$ and $\lim_{t \rightarrow \infty} N(x, x, y, t) = (1 + i_2)e^{i_1\theta}$, $\forall x, y \in X, t \in (0,1)$, where $a * b * c = abc$.

Let ϑ and Ω be self maps, defined as

$$\vartheta x = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 2, & \text{if } 2 < x \leq 5 \end{cases}, \quad \text{and } \Omega x = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 12, & \text{if } 2 < x \leq 5 \\ \frac{x+1}{3}, & \text{if } x > 5 \end{cases}$$

Let sequence $\{x_n\}$ be defined as $x_n = 5 + \frac{1}{n}, n \geq 1$

Then we have $\lim_{n \rightarrow \infty} \vartheta x = \lim_{n \rightarrow \infty} \Omega x = 2$. Hence, ϑ and Ω satisfy the property (E.A.).

and also, $\lim_{t \rightarrow \infty} N(\vartheta \Omega x_n, \vartheta \Omega x_n, \Omega \vartheta x_n, t) = \frac{t(1+i_2)e^{i_1\theta}}{t+|2-2|+|2-2|} = (1+i_2)e^{i_1\theta}$.

Thus these shows ϑ and Ω are compatible.

4. Application: In this article, motivated and inspired by the work of Ramot et al.[27] and Choi et al. [4]. We extend and improve some existing generalize the bicomplex valued N -fuzzy metric spaces by Singh et al.[25]. In order to demonstrate the usefulness of our established result in locating the unique solution to higher degree polynomial equations, we also offer an application. In the actual world, people employ polynomials to graph curves because they may be utilized to describe a variety of curve types. For example, a civil engineer would use polynomials to texture and style roads, buildings, and other structures, while an engineer designing a roller coaster would have used them to model the curves. For example, the designers of roller coasters might use polynomials to explain the curves in their rides. An example of a higher degree polynomial equation is one that has a degree of three or more. We have explained these higher degree polynomial equations in fuzzy metric spaces with bicomplex values.

For any natural number $\alpha \geq 3$ and real $|\beta| \leq 1$, the following equation $\beta^\alpha + 1 = (\alpha^4 - 1)\beta^\alpha + 1 + \alpha^4\beta$, has a unique real solution.

Proof: if $|\alpha| > 1$, the equation does not have a solution. So, let $X = \epsilon_2([0,1], R)$ and for all $\alpha, r \in X$,

let $N(\beta, \beta, r, t) = (1 + i_2)e^{i_1\theta} \frac{t}{|\beta-r|+|\beta-r|}$. Hence $(X, \mathcal{N}, *)$ is a complete bicomplex valued fuzzy metric spaces.

Now, let $\tilde{\mathfrak{F}}\beta = \frac{\beta^{\alpha+1}}{(\alpha^6-1)\beta^\alpha + \alpha^6}$

Notice that, since $\alpha \geq 2$, we can deduce that $\alpha^6 \geq 6$, for all $t > 0$ and $k \in (0,1)$.

Thus, $\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) = (1 + i_2)e^{i_1\theta} \frac{kt}{kt + |\mathfrak{F}\beta - \mathfrak{F}r| + |\mathfrak{F}\beta - \mathfrak{F}r|}$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) = (1 + i_2)e^{i_1\theta} \frac{kt}{kt + \left| \frac{\beta^{\alpha+1}}{(\alpha^6-1)\beta^{\alpha} + \alpha^6} - \frac{r^{\alpha+1}}{(\alpha^6-1)r^{\alpha} + \alpha^6} \right| + \left| \frac{\beta^{\alpha+1}}{(\alpha^6-1)\beta^{\alpha} + \alpha^6} - \frac{r^{\alpha+1}}{(\alpha^6-1)r^{\alpha} + \alpha^6} \right|}$$

$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} (1$

$$+ i_2)e^{i_1\theta} \frac{kt}{kt + \left| \frac{\beta^{\alpha} - r^{\alpha}}{((\alpha^6 - 1)\beta^{\alpha} + \alpha^6)((\alpha^6 - 1)r^{\alpha} + \alpha^6)} \right| + \left| \frac{\beta^{\alpha} - r^{\alpha}}{((\alpha^6 - 1)\beta^{\alpha} + \alpha^6)((\alpha^6 - 1)r^{\alpha} + \alpha^6)} \right|}$$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} (1 + i_2)e^{i_1\theta} \frac{kt}{kt + \left| \frac{\beta - \alpha}{\alpha^4} \right| + \left| \frac{\beta - \alpha}{\alpha^4} \right|}$$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} (1 + i_2)e^{i_1\theta} \frac{t}{t + \left| \frac{\beta - \alpha}{\alpha^4} \right| + \left| \frac{\beta - \alpha}{\alpha^4} \right|}$$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} (1 + i_2)e^{i_1\theta} \frac{t}{t + \left| \frac{\beta - \alpha}{\alpha} \right| + \left| \frac{\beta - \alpha}{\alpha} \right|}$$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} (1 + i_2)e^{i_1\theta} \frac{t}{t + |\beta - \alpha| + |\beta - \alpha|}$$

$$\mathcal{N}(\mathfrak{F}\beta, \mathfrak{F}\beta, \mathfrak{F}r, kt) \succeq_{i_2} \mathcal{N}(\beta, \beta, r, t)$$

As a result, the above theorem's requirements are met. The aforementioned equation thus has exactly one actual solution.

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