

## The invention of $\varepsilon$ - chainable $S_b$ -metric space with application

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### Abstract

In present paper, we introduce the notion of  $\varepsilon$ -chain in  $S_b$ -metric space and provide some supporting examples. As an application of  $\varepsilon$ -chain we prove a result of fixed point using Hardy Rogers[8] contraction condition. Our result generalizes and extends the theorem 3.1 of Pagey and Malviya [9] in  $S_b$ -metric Space[10].

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**Key words:** -  $\varepsilon$ -chain,  $\varepsilon$ -chainable  $S_b$ -metric space, Chainable  $S_b$ -metric space and Fixed point theorem.

### 1- Introduction and Preliminaries

In the history of topology  $\varepsilon$ -chain plays a very important role in defining connectedness in metric space. Firstly in 1883 Cantor defined connectedness with the help of  $\varepsilon$ -chains which have been studied extensively by many mathematicians [see Reference 1,2,3,4,5,6,7].

As long back Hardy Rogers[8] defined a generalization of contraction map and proved fixed point theorem. In 2009, Pagey and Malviya [9] proved many fixed point theorems in  $\varepsilon$ -chainable metric Space. On the other hand. In 2012, Sedghi et al.[11] defined a generalization of G metric space which is called S- metric space. In 2016, Souayan N. et al. [13] introduced the  $S_b$ -metric space as a generalization of the b-metric space and S metric Space, and proved some fixed point results under different types of contractions in a complete  $S_b$ -metric space. In 2017, Rohen Y et al. [10] modified the definition of  $S_b$ -metric and prove some coupled common fixed point theorems in  $S_b$ -metric space.

After introduction and preliminaries in section-2 we have defined  $\varepsilon$ -chain in  $S_b$ -metric space with examples and proved a fixed point theorem using Hardy Rogers condition.

**Definition 1.1[10]:** Let  $X$  be a nonempty set and  $s \geq 1$  be a given real number. An  $S_b$ -metric on  $X$  is a function  $S_b: X^3 \rightarrow [0, \infty)$  that satisfies the following conditions for all  $x, y, z, a \in X$ .

1.  $S_b(x, y, z) = 0$  if and only if  $x = y = z$
2.  $S_b(x, y, z) \leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$

Then the function  $S_b$  is called an  $S_b$ -metric and the pair  $(X, S_b)$  is called an  $S_b$ -metric space.

**Definition 1.2[10].** Let  $(X, S_b)$  be a  $S_b$ -metric space then  $S_b$  is called symmetric if

$$S_b(x, x, y) = S_b(y, y, x) \quad \forall x, y \in X \text{ (Symmetric Property)}$$

For the definitions of convergence and Cauchy sequence in  $S_b$  metric space reader can refer [10,12 and 13]

**2.Main Results :**

**Definition 2.1:-** Let  $(X, S_b)$  be a  $S_b$ -metric space. An  $\epsilon$ - chain is a finite succession of points  $a_0, a_1, a_2, \dots, a_{n-1}, a_n$  in  $X$  such that  $S_b(a_{i-1}, a_{i-1}, a_i) \leq \epsilon$  for  $i = 1, 2, \dots, n$ . The integer  $n$  is called the length of the  $\epsilon$ - chain

**Definition 2.2 :-** A space  $(X, S_b)$  is  $\epsilon$ -chainable ( $\epsilon$ -connected) if every pair of points in it can be joined by an  $\epsilon$ -chain of points in the set  $X$  and  $(X, S_b)$  is called chainable if it is  $\epsilon$ -chainable for each positive  $\epsilon$ .

The space  $(X, S_b)$  is called complete chainable  $S_b$ -metric space if every Cauchy sequence converges in it.

Throughout this paper length of  $\epsilon$ -chain between any two points  $x$  and  $y$  in  $X$  means length of shortest  $\epsilon$ -chain between points  $x$  and  $y$  in  $X$ .

**Example 2.1**  $X=A \cup B$  and  $S_b$  be the metric on  $X \times X \times X$  to  $R^+$  defined by

$$S_b(x, y, z) = (|x - z| + |y - z|)^2 \text{ where}$$

$$A = \{1, 1/2, 1/3\} \text{ and } B = \{1/2^2, 1/3^2, 1/4^2\}$$

Then  $Y$  is  $\epsilon$ -chainable for  $\epsilon = 1$  and length of biggest  $\epsilon$ -chain in  $X$  is 6.

**Example 2.2 :** Let  $(R, S_b)$  be the  $S_b$ - metric space as defined in Example 2.1 and  $Y = A \cup B$  where  $A = [1, 2)$  and  $B = (2, 3]$  Then  $(Y, d)$  is a disconnected subspace of  $(R, d)$  but is  $\epsilon$ -chainable for every  $\epsilon > 0$ . If  $\epsilon = 1/4$  then length of biggest  $\epsilon$ -chain in space  $(Y, d)$  is 5.

**Theorem 2.1:** Let  $(X, S_b)$  be a complete  $\epsilon$ -chainable symmetric  $S_b$ -metric space and let  $f, g$  be two self maps of  $X$ , if

$$\begin{aligned} \epsilon &= S_b(fx, fx, gy) \\ &\leq a_1 S_b(x, x, fx) + a_2 S_b(y, y, gy) + a_3 S_b(x, x, gy) + a_4 S_b(y, y, fx) \\ &\quad + a_5 S_b(x, x, y) \text{---(1)} \end{aligned}$$

for all  $x, y \in X$  where  $a_1, a_2, a_3, a_4, a_5$  are non negative reals. And

$$M s(a_1 + a_2 + a_3 + 3a_4 + a_5) < 1 \dots \dots \dots (2)$$

where  $M$  is the length of the biggest  $\varepsilon$ -chain in space  $(X, S)$  such that  $M \geq 0$ . Then  $f$  and  $g$  have unique common fixed point.

**Proof.** We define a sequence  $\{x_n\}$  as follows

$$fx_{2n+1} = x_{2n} \text{ and } gx_{2n+2} = x_{2n+1} \text{ for } n = 0,1,2,3 - -$$

If  $x_{2n} = x_{2n+1} = x_{2n+2}$  for some  $n$ , then we see that  $x_{2n}$  is a fixed point of  $f$  and  $g$

therefore we suppose that no two consecutive terms of sequence  $\{x_n\}$  are equal. If  $N$  is the length of  $\varepsilon$ -chain between  $x_{2n+1}$  and  $x_{2n+2}$  then  $\varepsilon$ -chainability of  $(X, S_b)$  gives.

$$S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq N\varepsilon$$

$$\leq M\varepsilon \quad \text{as } M \geq N$$

$$= MS_b(fx_{2n+1}, fx_{2n+1}, gx_{2n+2}) \quad \text{using (1)}$$

$$\leq M\{a_1S_b(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + a_2S_b(x_{2n+2}, x_{2n+2}, gx_{2n+2})$$

$$+ a_3S_b(x_{2n+1}, x_{2n+1}, gx_{2n+2}) + a_4S_b(x_{2n+2}, x_{2n+2}, fx_{2n+1}) + a_5S_b(x_{2n+1}, x_{2n+1}, x_{2n+2})\}$$

$$= M\{a_1S_b(x_{2n+1}, x_{2n+1}, x_{2n}) + a_2S_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + a_3S_b(x_{2n+1}, x_{2n+1}, x_{2n+1}) + a_4S_b(x_{2n+2}, x_{2n+2}, x_{2n}) + a_5S_b(x_{2n+1}, x_{2n+1}, x_{2n+2})\}$$

$$= M\{a_1S_b(x_{2n}, x_{2n}, x_{2n+1}) + (a_2 +$$

$$a_5)S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) + a_4S_b(x_{2n+2}, x_{2n+2}, x_{2n})\}$$

(by symmetric property and 1 of definition

1.1)

$$\leq M\{a_1S_b(x_{2n}, x_{2n}, x_{2n+1}) + (a_2 + a_5)S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) +$$

$$+ a_4[S_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_b(x_{2n}, x_{2n}, x_{2n+1})]\}$$

$$\leq M\{(a_1 + sa_4)S_b(x_{2n}, x_{2n}, x_{2n+1}) + (a_2 + a_5 + 2sa_4)S_b(x_{2n+1}, x_{2n+1}, x_{2n+2})\}$$

$$\Rightarrow [1 - M(a_2 + 2sa_4 + a_5)]S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq M\{(a_1 + sa_4)S_b(x_{2n}, x_{2n}, x_{2n+1})\}$$

$$\Rightarrow S_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) = \frac{M\{(a_1 + sa_4)\}}{[1 - M(a_2 + 2sa_4 + a_5)]} S_b(x_{2n}, x_{2n}, x_{2n+1})$$

In general

$$\Rightarrow S_b(x_n, x_n, x_{n+1}) = \frac{M(a_1 + sa_4)}{[1 - M(a_2 + 2sa_4 + a_5)]} S_b(x_{n-1}, x_{n-1}, x_n)$$

$$\Rightarrow S_b(x_n, x_n, x_{n+1}) = k S_b(x_{n-1}, x_{n-1}, x_n) \text{ where } k = \frac{M(a_1 + sa_4)}{[1 - M(a_2 + 2sa_4 + a_5)]}$$

$$\Rightarrow S_b(x_n, x_n, x_{n+1}) = k^n S_b(x_0, x_0, x_1) \text{ for } n = 1, 2, 3, \dots$$

Now we prove that  $\{x_n\}$  is a Cauchy sequence, if  $p$  is any positive integer then we have.

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sS_b(x_n, x_n, x_{n+1}) + sS_b(x_{n+p}, x_{n+p}, x_{n+1}) \text{ by 2 of definition 2.1}$$

$$= 2sS_b(x_n, x_n, x_{n+1}) + sS_b(x_{n+1}, x_{n+1}, x_{n+p}) \text{ by symmetric property}$$

$$\leq 2sS_b(x_n, x_n, x_{n+1}) + 2s^2S_b(x_{n+1}, x_{n+1}, x_{n+2}) + s^2S_b(x_{n+p}, x_{n+p}, x_{n+2})$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sS_b(x_n, x_n, x_{n+1}) + 2s^2S_b(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\dots \dots \dots + 2s^{n+p}S_b(x_{n+p-1}, x_{n+p-1}, x_{n+p})$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq 2sk^n S_b(x_0, x_0, x_1) + 2s^2k^{n+1} S_b(x_0, x_0, x_1)$$

$$\dots \dots \dots + 2s^{n+p}k^{n+p-1} S_b(x_0, x_0, x_1)$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) \leq (2sk^n + 2s^2k^{n+1} \dots \dots \dots + 2s^{n+p}k^{n+p-1})S_b(x_0, x_0, x_1)$$

$$\Rightarrow S_b(x_n, x_n, x_{n+p}) < \frac{2sk^n}{(1-sk)} S_b(x_0, x_0, x_1)$$

As  $n \rightarrow \infty$ ,  $S_b(x_n, x_n, x_{n+p}) \rightarrow 0$

Therefore  $\{x_n\}$  is a Cauchy sequence in  $X$ . As  $X$  is a complete space so there exists a point  $x$  in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = x$ .

**Existence of fixed point:**

Consider

$$S_b(x, x, gx) \leq 2sS_b(x, x, x_{2n}) + sS_b(x_{2n}, x_{2n}, gx) \text{ by 2 and symmetric property}$$

$$S_b(x, x, gx) \leq 2sS_b(x, x, x_{2n}) + sS_b(fx_{2n+1}, fx_{2n+1}, gx)$$

$$\leq 2sS_b(x, x, x_{2n}) + s\{a_1S_b(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + a_2S_b(x, x, gx) + a_3S_b(x_{2n+1}, x_{2n+1}, gx)\}$$

$$\begin{aligned}
 &+ a_4 S_b(x, x, f x_{2n+1}) + a_5 S_b(x_{2n+1}, x_{2n+1}, x) \} \\
 = &S_b(x, x, x_{2n}) + s\{a_1 S_b(x_{2n+1}, x_{2n+1}, x_{2n}) + a_2 S_b(x, x, g x) + a_3 S_b(x_{2n+1}, x_{2n+1}, g x) \\
 &+ a_4 S_b(x, x, x_{2n}) + a_5 S_b(x_{2n+1}, x_{2n+1}, x)\}
 \end{aligned}$$

As  $\{x_{2n+1}\}$  and  $\{x_{2n}\}$  are subsequences of  $\{x_n\}$ , as  $n \rightarrow \infty$   $x_{2n} \rightarrow x$  and  $x_{2n+1} \rightarrow x$

Therefore

$$\begin{aligned}
 S_b(x, x, g x) &\leq s(a_2 + a_3) S_b(x, x, g x) \\
 \Rightarrow [1 - s(a_2 + a_3)] S_b(x, x, g x) &\leq 0 \\
 \Rightarrow S_b(x, x, g x) = 0 &\text{ as } s(a_2 + a_3) < 1 \text{ by 2} \\
 \Rightarrow g x = x
 \end{aligned}$$

Similarly we can prove  $fx = x$ . This shows that  $x$  is common fixed point of  $f$  and  $g$ .

**Uniqueness:** Let  $u$  be another fixed point of  $f$  and  $g$

$$\begin{aligned}
 S_b(x, x, u) &= S_b(fx, fx, gu) \\
 &\leq a_1 S_b(x, x, fx) + a_2 S_b(u, u, gu) + a_3 S_b(x, x, gu) + a_4 S_b(u, u, fx) + a_5 S_b(x, x, u) \\
 &= a_1 S_b(x, x, x) + a_2 S_b(u, u, u) + a_3 S_b(x, x, u) + a_4 S_b(u, u, x) + a_5 S_b(x, x, u) \\
 &\leq (a_3 + a_4 + a_5) S_b(x, x, u) \text{ by symmetric property of } S_b \\
 \Rightarrow [1 - (a_3 + a_4 + a_5)] S_b(x, x, u) &\leq 0 \\
 \Rightarrow S_b(x, x, u) = 0 &\text{ as } (a_3 + a_4 + a_5) < 1 \\
 \Rightarrow x = u
 \end{aligned}$$

This complete the proof

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### References:

1. Beer, G., Which connected metric spaces are compact? Proc. Amer. Math. Soc. Volume 83, Number 4, 1981.
2. Bellamy, D.P., Short paths in homogeneous continua Topology Appl.26 (1987) 287-291.
3. Bing, R.H., Concerning hereditarily indecomposable continua, pacific J. Math. (1951) 43-51

4. Newman, M.H.A. Elements of Topology of plane sets of points Cambridge univ. Press, New york, 1961
5. MINC, P., A fixed point theorem for weakly chainable plane. Continua., transactions of the American. Mathematical Society, Volume 317, Number1 , January 1990.
6. Naimpally S.A and Pareek C.M., on the compactness of connected sets (preprint).
7. Kiran Shrivastava and Geeta Agrawal, Characterization of  $\varepsilon$ -chainable sets in metric spaces Indian j. Pure apl. Math 33(6): 933-940 June 2002.
8. Hardy, G.E. and Rogers T.D. “ A generalization of a fixed point theorem of Reich canod math. Bull 16 (1973)201-206.
9. Pagey S.S., Malviya N. “A chain of results on fixed point in an  $\varepsilon$  -- chainable metric space”“Journal of pure mathematics” University of Calcutta. Vol.26, , PP-67-78 , 2009.
10. Rohen Y., Dosenovic T., Radenovic S., A Note on the Paper “ A Fixed Point Theorems in  $S_b$ - Metric Space Filomat 31:11(2017) , 3335-3346 , <https://doi.org/10.2298/FIL1711335R>
11. Sedghi S, N. Shobe and A. Aliouche, A generalization of fixed point theo rems in S-metric spaces Mat. Vesnik 64,3(2012),258-266.
12. Sedghi S., Gholidahneh A. , Dosenovic T., Esfahani J, Radenovic S., Common Fixed Point of Four Maps in  $S_b$ - Metric Spaces, Journal of Linear and Topological Algebra , Vol 05, No. 02, 2016,93-104.
13. Souayan N, Mlaiki N., A Fixed Point Theorem in  $S_b$ - Metric Spaces, J.Math. Comput. Sci Vol 16 , Issue 2 ( 2016 ) , 131-139.<http://dx.doi.org/10.22436/jmcs.016.02.01>